

This is the definition of a Bézier curve. When $t = 0$ then $x = x_0$ and $y = y_0$.

$$x = a t^3 + b t^2 + c t + x_0$$

$$y = d t^3 + e t^2 + f t + y_0$$

When $t = 1$ then $x = x_3$ and $y = y_3$.

$$x_3 = a 1^3 + b 1^2 + c 1 + x_0$$

$$y_3 = d 1^3 + e 1^2 + f 1 + y_0$$

Now solve for a and d .

$$a = -b - c - x_0 + x_3, d = -e - f - y_0 + y_3$$

Substitute for a and d .

$$x = b t^2 (1-t) + c t (1-t^2) + t^3 (x_3 - x_0) + x_0$$

$$y = e t^2 (1-t) + f t (1-t^2) + t^3 (y_3 - y_0) + y_0$$

Find the slope of x and y .

$$\frac{dx}{dt} = b t (2 - 3t) + c (1 - 3t^2) + 3 t^2 (x_3 - x_0)$$

$$\frac{dy}{dt} = e t (2 - 3t) + f (1 - 3t^2) + 3 t^2 (y_3 - y_0)$$

Find the slope at the point $\{x_0, y_0\}$.

$$b 0 (2 - 3 * 0) + c (1 - 3 * 0^2) + 3 * 0^2 (x_3 - x_0) = c$$

$$e 0 (2 - 3 * 0) + f (1 - 3 * 0^2) + 3 * 0^2 (y_3 - y_0) = f$$

Point $\{x_1, y_1\}$ is on a line thru $\{x_0, y_0\}$ using the same slope and $t = 1/3$.

$$x_1 = \frac{c}{3} + x_0$$

$$y_1 = \frac{f}{3} + y_0$$

Solve for c and f .

$$c = 3 (x_1 - x_0), f = 3 (y_1 - y_0)$$

Substitute for c and f .

$$x = b t^2 (1-t) + t^3 (2 x_0 - 3 x_1 + x_3) + 3 t (x_1 - x_0) + x_0$$

$$y = e t^2 (1-t) + t^3 (2 y_0 - 3 y_1 + y_3) + 3 t (y_1 - y_0) + y_0$$

Find the slope of x and y .

$$\frac{dx}{dt} = b t (2 - 3t) + 3 t^2 (2 x_0 - 3 x_1 + x_3) + 3 (x_1 - x_0)$$

$$\frac{dy}{dt} = e t (2 - 3t) + 3 t^2 (2 y_0 - 3 y_1 + y_3) + 3 (y_1 - y_0)$$

Find the slope at the point $\{x_3, y_3\}$.

$$b 1 (2 - 3 * 1) + 3 * 1^2 (2 x_0 - 3 x_1 + x_3) + 3 (x_1 - x_0) = 3 (x_0 - 2 x_1 + x_3) - b$$

$$e \cdot 1(2 - 3 * 1) + 3 * 1^2(2 y_0 - 3 y_1 + y_3) + 3(y_1 - y_0) = 3(y_0 - 2 y_1 + y_3) - e$$

Point $\{x_2, y_2\}$ is on a line thru $\{x_3, y_3\}$ using the same slope and $t = 1/3$.

$$x_3 = \frac{3(x_0 - 2x_1 + x_3) - b}{3} + x_2$$

$$y_3 = \frac{3(y_0 - 2y_1 + y_3) - e}{3} + y_2$$

Solve for b and e .

$$b = 3(x_0 - 2x_1 + x_2), e = 3(y_0 - 2y_1 + y_2)$$

Substitute for b and e .

$$x = -t^3(x_0 - 3x_1 + 3x_2 - x_3) + 3t^2(x_0 - 2x_1 + x_2) + 3t(x_1 - x_0) + x_0$$

$$y = -t^3(y_0 - 3y_1 + 3y_2 - y_3) + 3t^2(y_0 - 2y_1 + y_2) + 3t(y_1 - y_0) + y_0$$

This is the Bézier curve in terms of the control points. $\{[x_0, y_0], [x_1, y_1], [x_2, y_2], [x_3, y_3]\}$.

Now I will simplify the calculations using a difference table. These are the first differences.

$$\begin{aligned} x_{01} &= x_1 - x_0 \\ y_{01} &= y_1 - y_0 \\ x_{12} &= x_2 - x_1 \\ y_{12} &= y_2 - y_1 \\ x_{23} &= x_3 - x_2 \\ y_{23} &= y_3 - y_2 \end{aligned}$$

Substitute in the first differences.

$$\begin{aligned} x &= t^3(x_{01} - 2x_{12} + x_{23}) + 3t^2(x_{12} - x_{01}) + 3t x_{01} + x_0 \\ y &= t^3(y_{01} - 2y_{12} + y_{23}) + 3t^2(y_{12} - y_{01}) + 3t y_{01} + y_0 \end{aligned}$$

These are the second differences.

$$\begin{aligned} x_{012} &= x_{12} - x_{01} \\ y_{012} &= y_{12} - y_{01} \\ x_{123} &= x_{23} - x_{12} \\ y_{123} &= y_{23} - y_{12} \end{aligned}$$

Substitute in the second differences.

$$\begin{aligned} x &= t^3(x_{123} - x_{012}) + 3t^2 x_{012} + 3t x_{01} + x_0 \\ y &= t^3(y_{123} - y_{012}) + 3t^2 y_{012} + 3t y_{01} + y_0 \end{aligned}$$

These are the third differences.

$$\begin{aligned} x_{0123} &= x_{123} - x_{012} \\ y_{0123} &= y_{123} - y_{012} \end{aligned}$$

Substitute in the third differences.

$$\begin{aligned} x &= t^3 x_{0123} + 3t^2 x_{012} + 3t x_{01} + x_0 \\ y &= t^3 y_{0123} + 3t^2 y_{012} + 3t y_{01} + y_0 \end{aligned}$$

The fastest way to evaluate a Bézier curve is the following:

$$\begin{aligned} b &= x_{012} + x_{012} + x_{012} \\ c &= x_{01} + x_{01} + x_{01} \\ e &= y_{012} + y_{012} + y_{012} \\ f &= y_{01} + y_{01} + y_{01} \\ x &= ((x_{0123}t + b)t + c)t + x_0 \\ y &= ((y_{0123}t + e)t + f)t + y_0 \end{aligned}$$

The following is the TeX source for the previous equations. Tex is not as easy to use as PostScript, but I can not do equations in PostScript yet. I hope to work on a project to do equations in PostScript unless somebody has put equations in PostScript.

This is the definition of a Bézier curve. When $t=0$,
then $x=x_0$, and, $y=y_0$.

$\$x = a, \t^3 + b, \t^2 + c, \t + x_0$
 $\$y = d, \t^3 + e, \t^2 + f, \t + y_0$

When $t=1$, then $x=x_3$, and, $y=y_3$.

$\$x_3 = a, 1^3 + b, 1^2 + c, 1 + x_0$
 $\$y_3 = d, 1^3 + e, 1^2 + f, 1 + y_0$

Now solve for a and d .

$\$a = -b - c - x_0 + x_3, \quad d = -e - f - y_0 + y_3$

Substitute for a and d .

$\$x = b, \t^2 \left(1 - t^2 \right) + c, \t \left(1 - t^2 \right) + t^3 \left(x_3 - x_0 \right) + x_0$
 $\$y = e, \t^2 \left(1 - t^2 \right) + f, \t \left(1 - t^2 \right) + t^3 \left(y_3 - y_0 \right) + y_0$

Find the slope of x and y .

$\$ \{ dx / dt \} = b, \t \left(2 - 3t \right) + c \left(1 - 3t^2 \right) + 3, \t^2 \left(x_3 - x_0 \right)$
 $\$ \{ dy / dt \} = e, \t \left(2 - 3t \right) + f \left(1 - 3t^2 \right) + 3, \t^2 \left(y_3 - y_0 \right)$

Find the slope at the point (x_0, y_0) .

$\$b, 0 \left(2 - 3^0 \right) + c \left(1 - 3^0 \right) + 3^0 \left(x_3 - x_0 \right) = c$
 $\$e, 0 \left(2 - 3^0 \right) + f \left(1 - 3^0 \right) + 3^0 \left(y_3 - y_0 \right) = f$

Point (x_1, y_1) is on a line thru (x_0, y_0) using the same slope and $t=1/3$.

$\$x_1 = \frac{c}{3} + x_0$
 $\$y_1 = \frac{f}{3} + y_0$

Solve for c and f .

$\$c = 3 \left(x_1 - x_0 \right)$, $f = 3 \left(y_1 - y_0 \right)$

Substitute for c and f .

$\$x = b, \t^2 \left(1 - t^2 \right) + t^3 \left(2, x_0 - 3, x_1 + x_3 \right) + 3, \t \left(x_1 - x_0 \right) + x_0$
 $\$y = e, \t^2 \left(1 - t^2 \right) + t^3 \left(2, y_0 - 3, y_1 + y_3 \right) + 3, \t \left(y_1 - y_0 \right) + y_0$

Find the slope of x and y .

$\$ \{ dx / dt \} = b, \t \left(2 - 3t \right) + 3, \t^2 \left(2, x_0 - 3, x_1 + x_3 \right) + 3 \left(x_1 - x_0 \right)$
 $\$ \{ dy / dt \} = e, \t \left(2 - 3t \right) + 3, \t^2 \left(2, y_0 - 3, y_1 + y_3 \right) + 3 \left(y_1 - y_0 \right)$

Find the slope at the point (x_3, y_3) .

$\$b, 1 \left(2 - 3^1 \right) + 3^1 \left(2, x_0 - 3, x_1 + x_3 \right) + 3 \left(x_1 - x_0 \right) = 3 \left(x_0 - 2, x_1 + x_3 \right) - b$
 $\$e, 1 \left(2 - 3^1 \right) + 3^1 \left(2, y_0 - 3, y_1 + y_3 \right) + 3 \left(y_1 - y_0 \right) = 3 \left(y_0 - 2, y_1 + y_3 \right)$

$-2x_1 + y_3)$ -e\$\$

Point $\{x_2, y_2\}$ is on a line thru $\{x_3, y_3\}$ \$ using the same slope and $t=1/3$.

$\$x_3 = \{3(x_0 - 2x_1 + x_3) / 3 + x_2\}$$$

$\$y_3 = \{3(y_0 - 2y_1 + y_3) / 3 + y_2\}$$$

Solve for b and e .

$\$b = 3(x_0 - 2x_1 + x_2) / 3, e = 3(y_0 - 2y_1 + y_2) / 3$

Substitute for b and e .

$\$x = -t^3(x_0 - 3x_1 + 3x_2 - x_3) / 3 + t^2(x_0 - 2x_1 + x_2) / 3 + t(x_1 - x_0) / 3$

$\$y = -t^3(y_0 - 3y_1 + 3y_2 - y_3) / 3 + t^2(y_0 - 2y_1 + y_2) / 3 + t(y_1 - y_0) / 3$

This is the B'ezier curve in terms of the control points. $\{x_0, y_0\}, \{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}$.

Now I will simplify the calculations using a difference table.

These are the first differences.

$\$x_{01} = x_1 - x_0$

$\$y_{01} = y_1 - y_0$

$\$x_{12} = x_2 - x_1$

$\$y_{12} = y_2 - y_1$

$\$x_{23} = x_3 - x_2$

$\$y_{23} = y_3 - y_2$

Substitute in the first differences.

$\$x = t^3(x_{01} - 2x_{12} + x_{23}) / 3 + t^2(x_{12} - x_{01}) / 3 + t(x_{01}) / 3$

$\$y = t^3(y_{01} - 2y_{12} + y_{23}) / 3 + t^2(y_{12} - y_{01}) / 3 + t(y_{01}) / 3$

These are the second differences.

$\$x_{012} = x_{12} - x_{01}$

$\$y_{012} = y_{12} - y_{01}$

$\$x_{123} = x_{23} - x_{12}$

$\$y_{123} = y_{23} - y_{12}$

Substitute in the second differences.

$\$x = t^3(x_{012} - x_{123}) / 3 + t^2(x_{123} - x_{012}) / 3 + t(x_{012}) / 3$

$\$y = t^3(y_{012} - y_{123}) / 3 + t^2(y_{123} - y_{012}) / 3 + t(y_{012}) / 3$

These are the third differences.

$\$x_{0123} = x_{123} - x_{012}$

$\$y_{0123} = y_{123} - y_{012}$

Substitute in the third differences.

$\$x = t^3(x_{0123} + x_{012}) / 3 + t^2(x_{012} - x_{0123}) / 3 + t(x_{0123}) / 3$

$\$y = t^3(y_{0123} + y_{012}) / 3 + t^2(y_{012} - y_{0123}) / 3 + t(y_{0123}) / 3$

The fastest way to evaluate a B'ezier curve is the following:

$\$b = x_{012} + x_{012} + x_{012}$

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 $$c =x_{01}+x_{01}+x_{01}$$  
 $$e =y_{012}+y_{012}+y_{012}$$  
 $$f =y_{01}+y_{01}+y_{01}$$  
 $$x =\left( \left( x_{0123},t+b \right) t+c \right) t+x_0$$  
 $$y =\left( \left( y_{0123},t+e \right) t+f \right) t+y_0$$  
 \bye
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