TEX source

The length of a Berier curve is an elliptic integral. This is a Bezier curve.

$$x = a_3\, t^3 + a_2\, t^2 + a_1\, t + x_0$$

$$y = b_3\, t^3 + b_2\, t^2 + b_1\, t + y_0$$

The control points of a Bezier curve are:

$$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2) \quad (x_3, y_3)$$

This is the curve for those control points.

$$a_1 = 3\,(x_1 - x_0) \quad a_2 = 3\,(x_0 - 2\,x_1 + x_2) \quad a_3 = -x_0 + 3\,x_1 - 3\,x_2 + x_3$$

$$b_1 = 3\,(y_1 - y_0) \quad b_2 = 3\,(y_0 - 2\,y_1 + y_2) \quad b_3 = -y_0 + 3\,y_1 - 3\,y_2 + y_3$$

The length of the Bezier cuvre form 0 to u is:

$$\int_0^u \sqrt{(a_1 + 2\,a_2\,t + 3\,a_3\,t^2)^2 + (b_1 + 2\,b_2\,t + 3\,b_3\,t^2)^2}\,dt$$

The control points of an example Bezier curve are:

$$(x_0 = 6, y_0 = 8) \quad (x_1 = 1, y_1 = 10) \quad (x_2 = 7, y_2 = 3) \quad (x_3 = 4, y_3 = 4)$$

Solve for $a_1, a_2, a_3, b_1, b_2, b_3$.

$$a_1 = -15 \quad a_2 = 33 \quad a_3 = -20 \quad b_1 = 6 \quad b_2 = -27 \quad b_3 = 17$$

The Bezier curve for this example is:

$$x = -20\,t^3 + 33\,t^2 - 15\,t + 6 \quad y = 17\,t^3 - 27\,t^2 + 6\,t + 8$$

For this example the length is:

$$3 \int_0^1 \sqrt{689\,t^4 - 1492\,t^3 + 1076\,t^2 - 292\,t + 29}\,dt$$

Numerical integration gives 7.237223368328592885826619210956022413967067986017504163362886516

DERIVE factored the polynomial:

$$689\,t^4 - 1492\,t^3 + 1076\,t^2 - 292\,t + 29 =$$

$$689\left(t + \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} + i\left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} + \frac{7}{689}\right)\right)$$

$$\left(t + \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} - i\left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} + \frac{7}{689}\right)\right)$$

$$\left(t - \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} + i\left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} - \frac{7}{689}\right)\right)$$

$$\left(t-\sqrt{\frac{\sqrt{6005}}{1378}+\frac{23377}{474721}}-\frac{373}{689}+i\left(\frac{7}{689}-\sqrt{\frac{\sqrt{6005}}{1378}-\frac{23377}{474721}}\right)\right)$$

Now define values k, m, a, b:

$$k=3\sqrt{689}\quad m=[1,1,1,1]\quad b=[1,1,1,1]\quad e=[e_1,e_2,e_3,e_4]$$

$$a=[-0.216589+0.0937743\,i,-0.216589-0.0937743\,i,-0.866139+0.0734550\,i,-0.866139-0.0734550\,i]$$

Do a partial fraction expansion.

$$I(m)=D219(1,m,4,a,b,e)$$

$$I(m)=I(4\,e_1)+(1.29909+0.375097\,i)\,I(3\,e_1)+(0.383342+0.365466\,i)\,I(2\,e_1)$$
$$-(0.0228475-0.0784923\,i)\,I(e_1)$$

Reduce $I(4\,e_1)$.

$$R35(0,1,4,a,b,e)=0$$

$$A(2\,e_1+e_2+e_3+e_4)+3\,I(4\,e_1)+(3.24774+0.937743\,i)\,I(3\,e_1)+(0.766684+0.730933\,i)\,I(2\,e_1)$$
$$-(0.0342712-0.117738\,i)\,I(e_1)=0$$

$$A(2\,e_1+e_2+e_3+e_4)=-0.138815+0.00794130\,i$$

Substitute $I(4\,e_1)$.

$$I(m)=(0.216516+0.0625162\,i)\,I(3\,e_1)+(0.127780+0.121822\,i)\,I(2\,e_1)$$
$$-(0.0114237-0.0392461\,i)\,I(e_1)+0.0462716-0.00264710\,i$$

Reduce $I(3\,e_1)$.

$$R35(-1,1,4,a,b,e)=0$$

$$A(e_1+e_2+e_3+e_4)+2\,I(3\,e_1)+(1.94864+0.562645\,i)\,I(2\,e_1)$$
$$+(0.383342+0.365466\,i)\,I(e_1)-(0.0114237-0.0392461\,i)\,I(0)=0$$

$$A(e_1+e_2+e_3+e_4)=-0.0846852$$

Substitute $I(3\,e_1)$.

$$I(m)=-0.0655894\,I(2\,e_1)-(0.0415000+0.0123012\,i)\,I(e_1)$$
$$+(0.00246348-0.00389163\,i)\,I(0)+0.0554395$$

Reduce $I(2\,e_1)$.

$$R35(-2,1,4,a,b,e)=0$$

$$A(e_2+e_3+e_4)+I(2\,e_1)+(0.0114237-0.0392461\,i)\,I(-e_1)$$
$$+(0.649549+0.187548\,i)\,I(e_1)=0$$

$$A(e_2+e_3+e_4)=-0.949300-0.327220\,i$$

Substitute $I(2\,e_1)$.

$$I(m)=(0.000749279-0.00257413\,i)\,I(-e_1)+0.00110358\,I(e_1)$$
$$+(0.00246348-0.00389163\,i)\,I(0)-0.00682448-0.0214622\,i$$

Calculate the basic integrals numerically.

$$I(-e_1)=-17.2709+33.5142\,i\quad I(0)=12.0414\quad I(e_1)=-3.86310-1.12917\,i$$

Substitute the values of the integrals.

$$I(m)=0.0919054\qquad k\,I(m)=7.23722$$

That is a good match.

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