

The way the Bézier curve is usually represented is:

$$\begin{aligned}x &= a t^3 + b t^2 + c t + x_0 \\y &= d t^3 + e t^2 + f t + y_0\end{aligned}$$

This does not represent the nature of the function very well. Both ends of the function are the same. Point $\{x_0, y_0\}$ can be exchanged with point $\{x_3, y_3\}$ without changing the function. A better way to represent the Bézier curve is in terms of variable u , where $u = 2 t - 1$. $u = -1$ when $t = 0$ and $u = +1$ when $t = 1$.

Now the Bézier curve is:

$$\begin{aligned}x &= a_3 u^3 + a_2 u^2 + a_1 u + a_0 \\y &= b_3 u^3 + b_2 u^2 + b_1 u + b_0\end{aligned}$$

$u = -1$ at point $\{x_0, y_0\}$ and $u = +1$ at point $\{x_3, y_3\}$.

$$\begin{aligned}x_0 &= a_3 (-1)^3 + a_2 (-1)^2 + a_1 (-1) + a_0 \\x_3 &= a_3 (+1)^3 + a_2 (+1)^2 + a_1 (+1) + a_0\end{aligned}$$

$$a_0 = \frac{x_3 + x_0}{2} - a_2$$

$$a_1 = \frac{x_3 - x_0}{2} - a_3 \tag{1}$$

We do not have to solve for the y values since they look just like the x values.

$$b_0 = \frac{y_3 + y_0}{2} - b_2$$

$$b_1 = \frac{y_3 - y_0}{2} - b_3$$

Two more control points are needed to find the curve. These control points set the slope and velocity at the end points of the curve.

$$\frac{dx}{du} = 3 a_3 u^2 + 2 a_2 u + a_1$$

When $t = \frac{1}{3}$ then $u = -\frac{1}{3}$ and when $t = \frac{2}{3}$ then $u = +\frac{1}{3}$.

When $u = -1$ then $x_1 = x_0 + \frac{dx}{du} \left(\frac{2}{3} \right)$.

When $u = +1$ then $x_3 = x_2 + \frac{dx}{du} \left(\frac{2}{3} \right)$.

$$x_1 = x_0 + (3 a_3 (-1)^2 + 2 a_2 (-1) + a_1) \left(\frac{2}{3} \right)$$

$$x_3 = x_2 + (3 a_3 (+1)^2 + 2 a_2 (+1) + a_1) \left(\frac{2}{3} \right)$$

$$\begin{aligned}
x_1 &= x_0 + 2 a_3 - \frac{4}{3} a_2 + \frac{2}{3} a_1 \\
x_3 &= x_2 + 2 a_3 + \frac{4}{3} a_2 + \frac{2}{3} a_1 \\
x_3 - x_1 &= x_2 - x_0 + \frac{8}{3} a_2 \\
a_2 &= \frac{3 (x_3 - x_2 - x_1 + x_0)}{8} \\
x_3 + x_1 &= x_2 + x_0 + 4 a_3 + \frac{4}{3} a_1
\end{aligned} \tag{2}$$

Combine equations (1) and (2).

$$\begin{aligned}
x_3 + x_1 &= x_2 + x_0 + 4 a_3 + \frac{4}{3} \left(\frac{x_3 - x_0}{2} - a_3 \right) \\
3 (x_3 + x_1 - x_2 - x_0) &= 12 a_3 + 2 (x_3 - x_0) - 4 a_3 \\
x_3 + 3 (x_1 - x_2) - x_0 &= 8 a_3 \\
a_3 &= \frac{x_3 + 3 (x_1 - x_2) - x_0}{8}
\end{aligned}$$

We do not have to solve for the y values since they look just like the x values.

$$\begin{aligned}
b_2 &= \frac{3 (y_3 - y_2 - y_1 + y_0)}{8} \\
b_3 &= \frac{y_3 + 3 (y_1 - y_2) - y_0}{8}
\end{aligned}$$

Now I solved for the Bézier curve only using Times-Roman and Times-Roman Italic. It was a lot harder to use WordPerfect than using TeX but I like the results better. The Bézier in terms of u is more like the actual curve than in terms of t .