How to determine the control points of a Bézier curve that approximates a small circular arc

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Bernstein polynomial

$$B_{i,n}(t)=\binom{n}{k}t^i(1-t)^{(n-i)}$$

Bézier-Bernstein curve

$$C(t)=\sum_{i=0}^{n} P_i B_{i,n}(t)$$

Consider a four point Bézier

$$\mathrm{C}(t)=(1-t)^3 P_0+3t(1-t)^2 P_1+3t^2(1-t)P_2+t^3 P_3 \qquad (1)$$

$$\mathrm{C}(t)=P_0+3(P_1-P_0)t+3(P_2-2P_1+P_0)t^2+(P_3-3P_2+3P_1-P_0)t^3$$

$$\mathrm{C}(0)=P_0$$

$$\mathrm{C}(1)=P_3$$

and its derivative

$$\mathrm{C}'(t)=-3(1-t)^2 P_0+3(1-t)(1-3t)P_1+3t(2-3t)P_2+3t^2 P_3 \qquad (2)$$

$$\mathrm{C}'(t)=3(P_1-P_0)+6(P_2-2P_1+P_0)t+3(P_3-3P_2+3P_1-P_0)t^2$$

$$\mathrm{C}'(0)=3(P_1-P_0)$$

$$\mathrm{C}'(1)=3(P_3-P_2)$$

indicate segment P_0P_1 is tanget to C at P_0, as is P_3P_2 at P_3.

Consider an arc A of sweep θ, < 90°, bisected by the x-axis. Let $\phi=\frac{\theta}{2}$

Force the endpoints of C to the endpoints of A. Note the symmetry due to bisection.

$$x_0=\cos(\phi) \qquad x_3=\cos(\phi) \qquad x_3=x_0 \qquad (3)$$

$$y_0=\sin(\phi) \qquad y_3=-\sin(\phi) \qquad y_3=-y_0$$

Force the midtime of C to the midpoint of A.

$$\mathrm{P}(\frac{1}{2})=(\frac{1}{2})^3 P_0+3(\frac{1}{2})(\frac{1}{2})^2 P_1+3(\frac{1}{2})^2(\frac{1}{2})P_2+P_3=(1{,}0) \qquad (4)$$

Substitute x_3 with x_0 and y_3 with $-y_0$

$$\frac{1}{8}x_0+\frac{3}{8}x_1+\frac{3}{8}x_2+\frac{1}{8}x_3=1 \qquad \frac{1}{8}y_0+\frac{3}{8}y_1+\frac{3}{8}y_2+\frac{1}{8}y_3=0$$

$$2x_0+3x_1+3x_2=8 \qquad 3y_1+3y_2=0 \qquad (5)$$

$$3x_2=8-2x_0-3x_1 \qquad y_2=-y_1$$

Force slopes of end point tangents of C to coincide with those of A .

$$m_0=\frac{y_0}{x_0} \quad m_0^t=\frac{-x_0}{y_0}=\frac{(y_0-y_1)}{(x_0-x_1)} \quad -x_0^2+x_0x_1=y_0^2-y_0y_1 \quad x_0x_1=x_0^2+y_0^2-y_0y_1=1-y_0y_1 \tag{6}$$

$$m_3=\frac{y_3}{x_3} \quad m_3^t=\frac{-x_3}{y_3}=\frac{(y_3-y_2)}{(x_3-x_2)} \quad -x_3^2+x_2x_3=y_3^2-y_2y_3 \quad x_2x_3=x_3^2+y_3^2-y_2y_3=1-y_2y_3 \tag{7}$$

In (7) substitute $x_3 \text{ with } x_0, y_3 \text{ with} -y_0, \text{ and } y_2 \text{ with} -y_1$ and contrast with (6)

$x_0x_2=1-y_0y_1=x_0x_1,$ thus $x_1=x_2$ and substituting in (5) gives $x_1=\frac{4-x_0}{3}$

In (6) substitute x_1 and multiply through by 3

$$x_0(4-x_0)=3-3y_0y_1$$

$$y_1=\frac{3-x_0(4-x_0)}{3y_0}=\frac{(1-x_0)(3-x_0)}{3y_0}$$

In final:

$$x_0=\cos(\frac{\theta}{2}) \quad y_0=\sin(\frac{\theta}{2})$$

$$x_3=x_1 \quad y_3=-y_0$$

$$x_1=\frac{4-x_0}{3} \quad y_1=\frac{(1-x_0)(3-x_0)}{3y_0}$$

$$x_2=x_1 \quad y_2=-y_1$$

Use rotation, scaling and translation transformations on P to make the Bezier curve approximate a circular arc of sweep θ of an arbitrarily positioned and sized circle.