The problem is to draw a circle using only the curveto. For most circles four curves are required. For a circle with a radius of 234 points the maximum error is 0.2657 pixel at 300 bpi.

The Bézier curve in equation space is:

$$x = a\ t^3 + b\ t^2 + c\ t + x_0$$

$$x - x_0 = a\ t^3 + b\ t^2 + c\ t$$

When $t = 1$ then $x = x_3$.

$$x_3 - x_0 = a\ 1^3 + b\ 1^2 + c\ 1$$

Solve for c.

$$c = -a - b - x_0 + x_3$$

The equation for the circle is:

$$r^2 = (x - x_c)^2 + (y - y_c)^2$$

The center of the circle is at the point $[x_c, y_c]$.

$$[t = 0,\ x = x_0,\ y = y_0,\ y_c = y_0]$$

$$r^2 = (x_0 - x_c)^2 + (y_0 - y_0)^2$$

Solve for x_c.

$$x_c = x_0 - r$$

$$0 = x^2 + 2\ x\,(r - x_0) + y^2 - 2\ y_0\ y - 2\ r\ x_0 + x_0^2 + y_0^2$$

$$x - x_0 = a\ t^3 + b\ t^2 + (-a - b - x_0 + x_3)\,t$$

$$x - x_0 = t\left(a\left(t^2 - 1\right) + b\,(t - 1) - x_0 + x_3\right)$$

The Bézier curve in terms of x_3 is:

$$x + t\,(x_0 - x_3) - x_0 = t\left(a\left(t^2 - 1\right) + b\,(t - 1)\right)$$

$$t\,(x_0 - x_3) + x - x_0 = t\,(a\ t + a + b)\,(t - 1)$$

$$x = a\ t\,(t - 1)\,(t + 1) + b\ t\,(t - 1) + t\,(x_3 - x_0) + x_0$$

The slope of the Bézier curve is:

$$\frac{dx}{dt} = a\left(3\ t^2 - 1\right) + b\,(2\ t - 1) - x_0 + x_3$$

$$3\,(x_1 - x_0) = -a - b - x_0 + x_3$$

$$3\,(x_3 - x_2) = 2\ a + b - x_0 + x_3$$

$$a = -x_0 + 3\ x_1 - 3\ x_2 + x_3$$

$$b = 3\,(x_0 - 2\ x_1 + x_2)$$

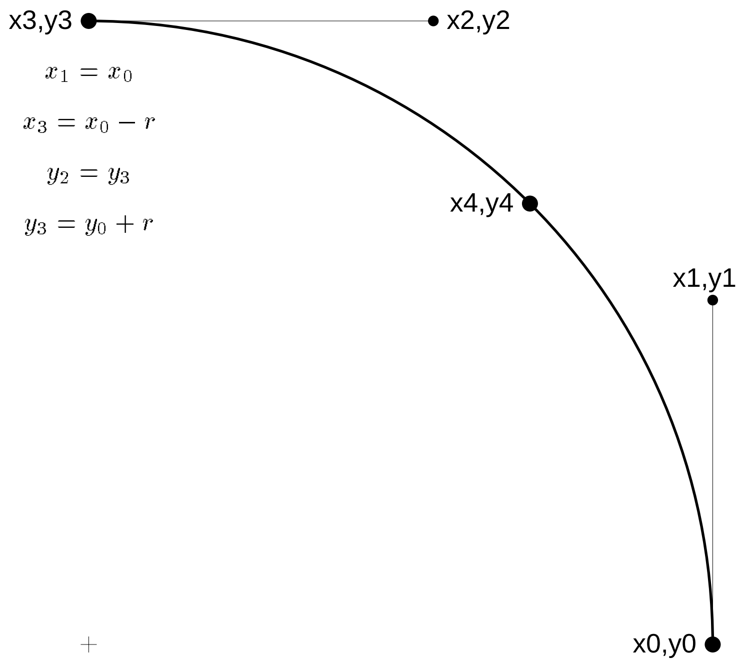
The Bézier curve in terms of the control points is:

$$x = (-x_0 + 3\ x_1 - 3\ x_2 + x_3)\,t\,(t - 1)\,(t + 1) + 3\,(x_0 - 2\ x_1 + x_2)\,t\,(t - 1) + t\,(x_3 - x_0) + x_0$$

$$x = -t^3\,(x_0 - 3\ x_1 + 3\ x_2 - x_3) + 3\ t^2\,(x_0 - 2\ x_1 + x_2) + 3\ t\,(x_1 - x_0) + x_0$$

$$y = -t^3\,(y_0 - 3\ y_1 + 3\ y_2 - y_3) + 3\ t^2\,(y_0 - 2\ y_1 + y_2) + 3\ t\,(y_1 - y_0) + y_0$$

For the circle the following is true:



$$x_1 = x_0$$

$$x_3 = x_0 - r$$

$$y_2 = y_3$$

$$y_3 = y_0 + r$$

The Bézier curve for the circle is:

$$x = -r\ t^3 + 3\ t^3\,(x_0 - x_2) + 3\ t^2\,(x_2 - x_0) + x_0$$

$$y = r\ t^2\,(3 - 2\ t) + 3\ t^3\,(y_1 - y_0) + 6\ t^2\,(y_0 - y_1) + 3\ t\,(y_1 - y_0) + y_0$$

We need more information to fit the curve to the circle, so let the curve and the circle touch halfway around the arc. Also let the curve be symmetrical. The point halfway around is $[x_4, y_4]$.

$$x_4 := x_0 - r + r\ \sin\left(\frac{\pi}{4}\right)$$

$$y_4 := y_0 + r\ \sin\left(\frac{\pi}{4}\right)$$

$$x_4 = -r\ t_4^3 + 3\ t_4^3\,(x_0 - x_2) + 3\ t_4^2\,(x_2 - x_0) + x_0$$

$$x_2 = \sqrt{2}\frac{\left(r\left(\sqrt{2}\,t_4^3 - \sqrt{2} + 1\right) + 3\sqrt{2}\,t_4^2\,x_0\,(1 - t_4)\right)}{(6\,t_4^2\,(1 - t_4))}$$

$$y_4 = r\ t_4^2\,(3 - 2\ t_4) + 3\ t_4^3\,(y_1 - y_0) + 6\ t_4^2\,(y_0 - y_1) + 3\ t_4\,(y_1 - y_0) + y_0$$

$$y_1 = \sqrt{2}\frac{\left(r\left(2\sqrt{2}\,t_4^3 - 3\sqrt{2}\,t_4^2 + 1\right) + 3\sqrt{2}\,t_4\,y_0\left(t_4^2 - 2\,t_4 + 1\right)\right)}{(6\,t_4\,(t_4^2 - 2\,t_4 + 1))}$$

Since the curve is symmetrical the following is true.

$$y_1 - y_0 = x_2 - x_3$$

$$y_1 - y_0 = x_2 - (x_0 - r)$$

$$y_1 = r - x_0 + x_2 + y_0$$

$$x_2 = \sqrt{2}\frac{\left(3\sqrt{2}\,t_4\,x_0\left(t_4^2 - 2\,t_4 + 1\right) - r\left(\sqrt{2}\,t_4^3 - 3\sqrt{2}\,t_4^2 + 3\sqrt{2}\,t_4 - 1\right)\right)}{(6\,t_4\,(t_4^2 - 2\,t_4 + 1))}$$

$$0 = \sqrt{2}\frac{r\left(2\sqrt{2}\,t_4^3 - 3\sqrt{2}\,t_4^2 + \sqrt{2}\,t_4\left(\sqrt{2} - 1\right) + \sqrt{2} - 1\right)}{(6\,t_4^2\,(t_4^2 - 2\,t_4 + 1))}$$

Solve for t.

$$t_4 = \frac{1}{2}$$

Solve for x_2 and y_1.

$$x_2 = \frac{r\left(4\sqrt{2}-7\right)}{3} + x_0$$

$$y_1 = \frac{4\,r\left(\sqrt{2}-1\right)}{3} + y_0$$

The equation of the curve is:

$$x = x_0 - r\,t^2\left(2\,t\left(2\sqrt{2}-3\right) - 4\sqrt{2}+7\right)$$

$$y = r\,t\left(2\,t^2\left(2\sqrt{2}-3\right) + t\left(11-8\sqrt{2}\right) + 4\sqrt{2}-4\right) + y_0$$

The relative error of the fit to the circle is e.

$$e = \frac{\sqrt{\left(x-x_0+r\right)^2 + \left(y-y_0\right)^2}}{r} - 1$$

The slope is zero at the maximum value of the error.

$$\frac{de}{dt} = 0$$

$$0 = t\left(34-24\sqrt{2}\right)\left(12\,t^4 - 30\,t^3 + 26\,t^2 - 9\,t + 1\right)$$

There are five solutions for t.

$$t = 0$$

$$t = \frac{1}{2}$$

$$t = 1$$

$$t = \frac{1}{2} - \frac{\sqrt{3}}{6}$$

$$t = \frac{\sqrt{3}}{6} + \frac{1}{2}$$

The maximum error is:

$$e = \sqrt{\frac{71}{54} - \frac{2\sqrt{2}}{9}} - 1$$

$$e = 2.7253000742770549017051 * 10^{-4}$$

This is an example of a large circle.

$$r = 234$$

$$x_0 = 539.8953$$

$$y_0 = 485.4492$$

$$x_2 = \frac{234\left(4\sqrt{2}-7\right)}{3} + 539.8953$$

$$x_2 = 435.1299314604056552261 26$$

$$y_1 = 4 * \frac{234\left(\sqrt{2}-1\right)}{3} + 485.4492$$

$$y_1 = 614.6838314604056552261 26$$

The error in pixels at 300 bpi is:

$$\frac{25\,e\,r}{6} = 0.2657167572420128529162 48$$