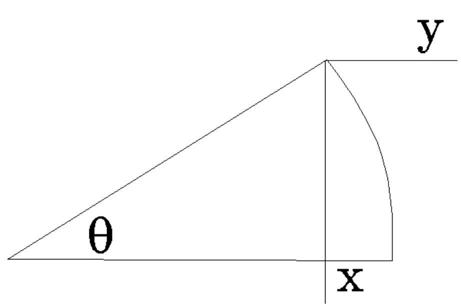
I am reading a book called Knot Insertion and Deletion Algorithms for B-Spline Curves and Surfaces. It is very hard to follow since it has few examples. I decided to do some examples of spline curves. The curves that will be used will be Bézier curves since they are widely used in graphics. I plan to use DERIVE a mathematical assistant for personal computers to do the algebra.

The first example is a simple arc of a circle.



The equation of the arc is $x = cos(\theta) \quad y = sin(\theta)$. We want the best fit from the Bézier curve $x = a\,t^3 + b\,t^2 + c\,t + d \quad y = e\,t^3 + f\,t^2 + g\,t + h$. This arc can be translated, scaled, and rotated to any position and size. Also the arc we are fitting can be part of a longer arc. If the best fit is not good enough then we can cut the arc in smaller arcs and fit each of those. Once we solve the problem for this arc we have solved the problem for any arc. Since this is a spline fit the best fit will be defined as having the minimum squared difference and matching exactly at the two end points. It must match slope and position at the end points. The control point form of the Bézier curve will be easier to handle for this kind of fit. The control points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$. The first end point is $x = x_0, y = y_0, t = 0$. Subsitute those values into the equation for the Bézier curve. The result is $x_0 = d, y_0 = h$. The equations are

$$x = a\,t^3 + b\,t^2 + c\,t + x_0 \quad y = e\,t^3 + f\,t^2 + g\,t + y_0$$

The other end point is $x = x_3, y = y_3, t = 1$. Subsitute those values into the equation for the Bézier curve. The result is

$$a = -b - c - x_0 + x_3 \quad e = -f - g - y_0 + y_3$$

The equations are

$$x = b\,t^2\,(1-t) + c\,t\,\left(1-t^2\right) + t^3\,(x_3 - x_0) + x_0$$

$$y = f\,t^2\,(1-t) + g\,t\,\left(1-t^2\right) + t^3\,(y_3 - y_0) + y_0$$

The first slope control points are x_1, y_1 at $t = 0$. They satisfy the equations

$$3\,(x_1 - x_0) = c \quad 3\,(y_1 - y_0) = g$$

The equations are

$$x = b\,t^2\,(1-t) + t^3\,(2\,x_0 - 3\,x_1 + x_3) + 3\,t\,(x_1 - x_0) + x_0$$

$$y = f\,t^2\,(1-t) + t^3\,(2\,y_0 - 3\,y_1 + y_3) + 3\,t\,(y_1 - y_0) + y_0$$

The second slope control points are x_2, y_2 at $t = 1$. They satisfy the equations

$$3\,(x_3 - x_2) = 3\,(x_0 - 2\,x_1 + x_3) - b$$

$$3\,(y_3 - y_2) = 3\,(y_0 - 2\,y_1 + y_3) - f$$

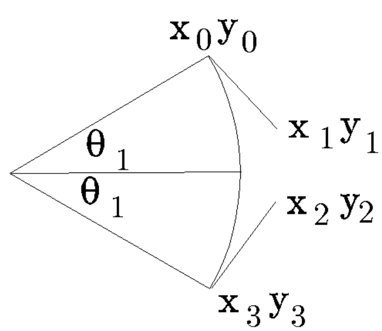
Solve for b and f.

$$b = 3\,(x_0 - 2\,x_1 + x_2)\,, f = 3\,(y_0 - 2\,y_1 + y_2)$$

The equations are

$$x = -t^3\,(x_0 - 3\,x_1 + 3\,x_2 - x_3) + 3\,t^2\,(x_0 - 2\,x_1 + x_2) + 3\,t\,(x_1 - x_0) + x_0$$

$$y = -t^3\,(y_0 - 3\,y_1 + 3\,y_2 - y_3) + 3\,t^2\,(y_0 - 2\,y_1 + y_2) + 3\,t\,(y_1 - y_0) + y_0$$



This arc is easy to model since it is symmetrical. We can see that $x_0 = cos\,(\theta_1)$, $y_0 = -sin\,(\theta_1)$, $x_3 = cos\,(\theta_1)$, $y_3 = sin\,(\theta_1)$. Substitute these into the Bézier curve. So now the Bézier curve is less general and passes through the end points.

$$x = \left(3\,t^2 - 3\,t + 1\right)\,cos\,(\theta_1) + 3\,t^3\,(x_1 - x_2) + 3\,t^2\,(x_2 - 2\,x_1) + 3\,t\,x_1$$

$$y = \left(2\,t^3 - 3\,t^2 + 3\,t - 1\right)\,sin\,(\theta_1) + 3\,t^3\,(y_1 - y_2) + 3\,t^2\,(y_2 - 2\,y_1) + 3\,t\,y_1$$

Since it is symmetrical $x_1 = x_2$ and $y_1 = -y_2$. Substitute these into the Bézier curve.

$$x = \left(3\,t^2 - 3\,t + 1\right)\,cos\,(\theta_1) - 3\,t^2\,x_1 + 3\,t\,x_1$$

$$y = \left(2\,t^3 - 3\,t^2 + 3\,t - 1\right)\,sin\,(\theta_1) + 6\,t^3\,y_1 - 9\,t^2\,y_1 + 3\,t\,y_1$$

The slope must match at two values of θ and t. This is the equation of the slopes from DERIVE. This is the slope of the Bézier curve.

$$\frac{dx}{dt} = (6\,t - 3)\,cos\,(\theta_1) - 6\,t\,x_1 + 3\,x_1$$

$$\frac{dy}{dt} = \left(6\,t^2 - 6\,t + 3\right)\ sin\,(\theta_1) + 18\,t^2\,y_1 - 18\,t\,y_1 + 3\,y_1$$

$$\frac{dx}{dy} = \frac{(2\,t - 1)\ (cos\,(\theta_1) - x_1)}{(2\,t^2 - 2\,t + 1)\ sin\,(\theta_1) + y_1\ (6\,t^2 - 6\,t + 1)}$$

This is the equation of the arc.

$$x = cos(\theta) \quad y = sin(\theta)$$

The slope of the arc is:

$$\frac{dx}{d\theta} = -sin(\theta) \quad \frac{dy}{d\theta} = cos(\theta)$$

$$\frac{dx}{dy} = -tan(\theta)$$

Set the slopes equal when $\theta = -\theta_1$ and $t = 0$.

$$tan\,(\theta_1) = \frac{x_1 - cos\,(\theta_1)}{sin\,(\theta_1) + y_1}$$

Solve for y_1.

$$y_1 = \frac{x_1\,cos\,(\theta_1) - 1}{sin\,(\theta_1)}$$

Subsitute this in the equations.

$$x = \left(3\,t^2 - 3\,t + 1\right)\ cos\,(\theta_1) - 3\,t^2\,x_1 + 3\,t\,x_1$$

$$y = \frac{\left(6\,t^3\,x_1 - 9\,t^2\,x_1 + 3\,t\,x_1\right)\ cos\,(\theta_1)}{sin\,(\theta_1)} +$$

$$\left(2\,t^3 - 3\,t^2 + 3\,t - 1\right)\ sin\,(\theta_1) - \frac{3\,t\ \left(2\,t^2 - 3\,t + 1\right)}{sin\,(\theta_1)}$$

To find the value of x_1 the curve will pass through another point on the arc. The point is $x = 1$, $y = 0$, and $t = t_1$. Substitute those values into the equations of the curve.

$$1 = \left(3\,t_1^2 - 3\,t_1 + 1\right)\,cos\left(\theta_1\right) - 3\,t_1^2\,x_1 + 3\,t_1\,x_1$$

$$0 = \frac{\left(6\,t_1^3\,x_1 - 9\,t_1^2\,x_1 + 3\,t_1\,x_1\right)\,cos\left(\theta_1\right)}{sin\left(\theta_1\right)} +$$

$$\left(2\,t_1^3 - 3\,t_1^2 + 3\,t_1 - 1\right)\,sin\left(\theta_1\right) - \frac{3\,t_1\,\left(2\,t_1^2 - 3\,t_1 + 1\right)}{sin\left(\theta_1\right)}$$

Solve for x_1.

$$x_1 = \frac{\left(3\,t_1^2 - 3\,t_1 + 1\right)\,cos\left(\theta_1\right) - 1}{3\,t_1\,\left(t_1 - 1\right)}$$

Substitute it into the previous equation.

$$0 = \frac{\left(2\,t_1 - 1\right)\,\left(\left(2\,t_1^2 - 2\,t_1\right)\,cos\left(\theta_1\right)^2 - cos\left(\theta_1\right) - 2\,t_1^2 + 2\,t_1 + 1\right)}{sin\left(\theta_1\right)}$$

Solve for t_1.

$$t_1 = \frac{1}{2}$$

Substitute the value of t_1 to get the final values of x_1 and y_1.

$$x_1 = \frac{4 - cos\left(\theta_1\right)}{3}$$

$$y_1 = \frac{\left(1 - cos\left(\theta_1\right)\right)\,\left(cos\left(\theta_1\right) - 3\right)}{3\,sin\left(\theta_1\right)}$$

Substitute x_1 and y_1 in the equations for the curve.

$$x - 1 = \left(2\,t - 1\right)^2\,\left(cos\left(\theta_1\right) - 1\right)$$

$$y = \frac{\left(1 - 2\,t\right)\,\left(cos\left(\theta_1\right) - 1\right)\,\left(\left(2\,t^2 - 2\,t + 1\right)\,cos\left(\theta_1\right) - 2\,t^2 + 2\,t + 1\right)}{sin\left(\theta_1\right)}$$