Finding the Length of a Bezier Cubic Spline and its Subdivisions

Don Lancaster
Synergetics, Box 809, Thatcher, AZ 85552

http://www.tinaja.com
don@tinaja.com
(928) 428-4073

Bezier **Cubic Splines** are an excellent and preferred method to draw the smooth continuous curves often found in typography, **CAD/CAM**, and graphics in general.

Among their many advantages is a **very sparse data set** allowing a mere **eight** values (or four **x,y** points) to **completely** define a full and carefully controlled and device independent curve. Many tutorials and examples are now present in our **Cubic Spline** Library. A brief and useful intro **appears here**. The fundamental math behind Cubic Splines **appears here**.

It is often of interest to try and find the exact **length** of a given cubic spline. First to find the total length itself, and secondarily to be able to **subdivide** the spline or find an exact point **along** its length for text positioning or other alignment.

It turns out that an **exact** closed form solution to the cubic spline length problem is **unbearably gruesome** as it involves **elliptic integrals** on top of possible cusp discontinuities and multiple values. Instead…

> **The exact length of a cubic spline is very difficult to find. Instead, for most uses, a chorded approximation is easier.**
>
> **Typically, 100 chords will suffice and will be accurate to a fraction of a percent or better.**

I've put together a set of utility routines as **BZLNSUB1.PSL** that will rapidly let you find spline lengths. In addition, you can also find the **accurate** x and y values for any **subdivided** position along the spline curve.

As with most of our utilities, this is written in raw **PostScript**, makes optional use of my **Gonzo Utilities**, and is used to create **standard ASCII text files** sent on to **Acrobat Distiller**. Used as a **General Purpose Host Based PostScript Interpreter**.

A crucial point before we continue…

The "t" parameter is in general nonlinear and tends to change faster along the "more bent" curve portions.

While a nonlinear "t" can be effectively used to find spline lengths, it most assuredly can NOT be used to find exact linear positions ALONG the spline curve.

The workaround is to create two linked and related arrays, one in "t" space and one in accumulated or integrated "s" space that can be interpolated.

Coincidentally, the very last entry in the "s" space accumulated length array will also be the spline length.

Thus, finding total spline lengths is fairly easy. Finding exact linear subdivision points in **"s"** space is considerably more difficult. But solve the latter and the total length pops out free. Which is the approach we will use here.

What follows is best understood if you have recently reviewed **CUBEZMATH.PDF** and have a copy of **BZLNSUB1.PSL** in an open window.

In general, a typical cubic spline is entered as **eight** data points **x0,y0** up through **x3,y3**. With the end points setting the **ends** of the curve, and the mid points determining the **slope and enthusiasm** (or tension) of the curve shape.

We can start with a few of the lower level service routines. A first order of business is relating **x0,y0** through **x3,y3** to the **A-H** parameters of the cubic spline equations themselves…

```
/findAH {
        /A x3 x2 3 mul sub x1 3 mul add x0 sub store
        /E y3 y2 3 mul sub y1 3 mul add y0 sub store
        /B x2 3 mul x1 6 mul sub x0 3 mul add  store
        /F y2 3 mul y1 6 mul sub y0 3 mul add  store
        /C x1 3 mul x0 3 mul sub store
        /G y1 3 mul y0 3 mul sub store
        /D x0 store    /H y0 store
      } store
```

Details on where these formulas come from are found in **CUBEMATH.PDF**.

Next are the "cubeless method" evaluation of **x** and **y** data points…

```
/findx {/ttt exch store A ttt mul B add ttt mul C add ttt
        mul D add /curx exch store} store

/findy {/ttt exch store E ttt mul F add ttt mul G add ttt
        mul H add /cury exch store} store
```

Along with a standard "**square root of the sum of the squares**" vector calculator to find the length of a chord given its present and previous **t** and **y** data values…

```
/finds {curx prevx sub dup mul cury prevy sub dup mul
        add sqrt /curs exch store} store
```

We need a method to create a **tlist**, which is simply a numerically ordered array of the **t** values. As in **/tlist [0 0.01 0.02 ... 0.99 1.00] store** . This is handled by a stock **/maketlist** routine in **BZLNSUB1.PSL** that accepts **numchords** as an input variable. As we've seen, a 100 chord approximation will often be acceptable.

Now for the Tricky Part...

You'll next want to create a **slist** companion to the **tlist**. In which the accumulated **s** distance appears for any given **t** value. For any **t** in the array, the new **x** and **y** values are compared against the previous ones and a new chord is found using **finds**. These values are then accumulated into the composite **slist**…

```
/makecurslist {
  /prevx 0 store            % initialize previous values
  /prevy 0 store
  /curslist mark  0         % start slist array
  0 1 numchords {curtlist exch get     % grab current t
        /tt exch store
        tt findx            % find curx and cury
        tt findy
        finds               % find current chord
        curs add dup        % integrate result; dup for next
        /prevx curx store       % save new chord end
        /prevy cury store
            } for
   pop ] store              % undo final sum, then store
         } store
```

Your total spline length is simply the last entry in your **slist** array…

```
/findsplinelen {curslist dup length 1 sub get
                    /splinelen exch store} store
```

And that's all you'll really need if your only interest is finding the spline length. Your final high level length-finding code can look something like this…

```
/findbezlen  {findAH        % calculate cubic coefficients
              makecurtlist  % create linear t array
              makecurslist  % create nonlinear s array
              findsplinelen % report length as splinelen
                } store
```

The **BZLNSUB1.PSL** utility then goes on to an example or two, noting length values of **31.2185** and **31.3125** and **31.3130** for **numchords** selections of **10**, **100**, and **1000**. Spline data of **0, 0, 0.01, 0.01, 19.9, 19.9, 20,20** has a known length of **28.28427** for which a 100 sample length returns **28.28430** . Giving us a good accuracy check.

Secrets of Subdivision

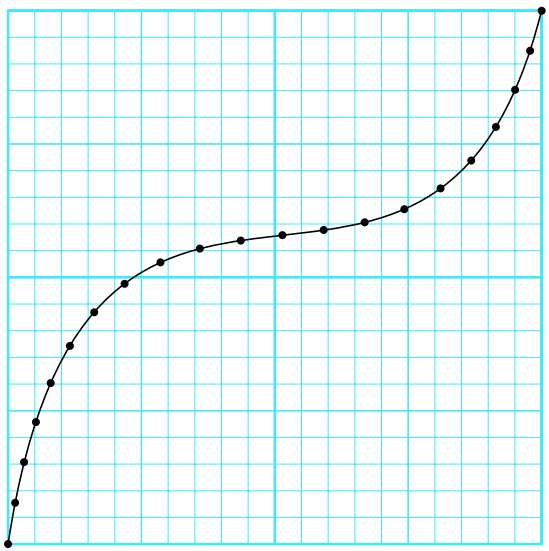
To find a **t** for a given **s**, you work your way through your **slist** until you find a "bin" that brackets the **s** you are after. Your entry fraction of that bin can then become your entry fraction on the companion **tlist** bin…

```
/findtofs { /ss exch store  % desired subdivision length
  1 1 curslist length 1 sub % scan each s value
    { /ii exch store
      ss curslist ii get    % check bins till too big
      le {exit} if } for    % then exit with high s
  /ii ii 1 sub store        % one less so s is boxed
  curslist ii 1 add get
  curslist ii get sub       % current s delta
  ss curslist ii get sub    % excess into interval
  exch div                  % interpolated s fraction
  curtlist ii 1 add get     % t delta
  curtlist ii get sub mul   % interpolated t fraction
  curtlist ii get add       % t for selected s
        } store
```

Your high level code to return **x**and **y** for a **normalized** 0-1 value for **s** is then…

```
/findxyofs {splinelen mul  % denormalize
        findtofs           % interpolated t for the LINEAR s
        dup findx          % get x(t)
        findy              % get y(t)
        curx cury          % and return to stack
        } store
```

Note that **findbezlen** must be run before using this routine. **BZLNSUB1.PSL** then winds things up with this example of a cubic spline that's subdivided into twenty **constant** "**s**" lengths …



For More Help

Additional info on cubic splines can be found on our **Cubic Spline** library page. As are many dozens of examples of Bezier cubic spline techniques.

Additional consulting services are available per our **Infopack** services and on a contract or an hourly basis. Additional **GuruGrams** are found **here**.

Further **GuruGrams** await your ongoing support as a **Synergetics Partner**.