This is the definition of a Bézier curve. When $t = 0\, then x = x_0\, and\, y = y_0$.

$$x = a\ t^3 + b\ t^2 + c\ t + x_0$$

$$y = d\ t^3 + e\ t^2 + f\ t + y_0$$

When $t = 1\, then x = x_3\, and\, y = y_3$.

$$x_3 = a\ 1^3 + b\ 1^2 + c\ 1 + x_0$$

$$y_3 = d\ 1^3 + e\ 1^2 + f\ 1 + y_0$$

Now solve for a and d.

$$a = -b - c - x_0 + x_3,\ d = -e - f - y_0 + y_3$$

Substitute for a and d.

$$x = b\ t^2\,(1 - t) + c\ t\left(1 - t^2\right) + t^3\,(x_3 - x_0) + x_0$$

$$y = e\ t^2\,(1 - t) + f\ t\left(1 - t^2\right) + t^3\,(y_3 - y_0) + y_0$$

Find the slope of x and y.

$$\frac{dx}{dt} = b\ t\,(2 - 3\,t) + c\left(1 - 3\,t^2\right) + 3\ t^2\,(x_3 - x_0)$$

$$\frac{dy}{dt} = e\ t\,(2 - 3\,t) + f\left(1 - 3\,t^2\right) + 3\ t^2\,(y_3 - y_0)$$

Find the slope at the point $\{x_0, y_0\}$.

$$b\ 0\,(2 - 3 * 0) + c\left(1 - 3 * 0^2\right) + 3 * 0^2\,(x_3 - x_0) = c$$

$$e\ 0\,(2 - 3 * 0) + f\left(1 - 3 * 0^2\right) + 3 * 0^2\,(y_3 - y_0) = f$$

Point $\{x_1, y_1\}$ is on a line thru $\{x_0, y_0\}$ using the same slope and $t = 1/3$.

$$x_1 = \frac{c}{3} + x_0$$

$$y_1 = \frac{f}{3} + y_0$$

Solve for c and f.

$$c = 3\,(x_1 - x_0)\,,\ f = 3\,(y_1 - y_0)$$

Substitute for c and f.

$$x = b\ t^2\,(1 - t) + t^3\,(2\,x_0 - 3\,x_1 + x_3) + 3\ t\,(x_1 - x_0) + x_0$$

$$y = e\ t^2\,(1 - t) + t^3\,(2\,y_0 - 3\,y_1 + y_3) + 3\ t\,(y_1 - y_0) + y_0$$

Find the slope of x and y.

$$\frac{dx}{dt} = b\ t\,(2 - 3\,t) + 3\ t^2\,(2\,x_0 - 3\,x_1 + x_3) + 3\,(x_1 - x_0)$$

$$\frac{dy}{dt} = e\ t\,(2 - 3\,t) + 3\ t^2\,(2\,y_0 - 3\,y_1 + y_3) + 3\,(y_1 - y_0)$$

Find the slope at the point $\{x_3, y_3\}$.

$$b\ 1\,(2 - 3 * 1) + 3 * 1^2\,(2\,x_0 - 3\,x_1 + x_3) + 3\,(x_1 - x_0) = 3\,(x_0 - 2\,x_1 + x_3) - b$$

$$e\ 1\,(2-3*1)+3*1^2\,(2\,y_0-3\,y_1+y_3)+3\,(y_1-y_0)=3\,(y_0-2\,y_1+y_3)-e$$

Point $\{x_2, y_2\}$ is on a line thru $\{x_3, y_3\}$ using the same slope and $t = 1/3$.

$$x_3=\frac{3\,(x_0-2\,x_1+x_3)-b}{3}+x_2$$

$$y_3=\frac{3\,(y_0-2\,y_1+y_3)-e}{3}+y_2$$

Solve for b and e.

$$b=3\,(x_0-2\,x_1+x_2)\,,\ e=3\,(y_0-2\,y_1+y_2)$$

Substitute for b and e.

$$x=-t^3\,(x_0-3\,x_1+3\,x_2-x_3)+3\ t^2\,(x_0-2\,x_1+x_2)+3\ t\,(x_1-x_0)+x_0$$

$$y=-t^3\,(y_0-3\,y_1+3\,y_2-y_3)+3\ t^2\,(y_0-2\,y_1+y_2)+3\ t\,(y_1-y_0)+y_0$$

This is the Bézier curve in terms of the control points. $[\{x_0, y_0\}, \{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}]$.

Now I will simplify the calculations using a difference table. These are the first differences.

$$x_{01}=x_1-x_0$$

$$y_{01}=y_1-y_0$$

$$x_{12}=x_2-x_1$$

$$y_{12}=y_2-y_1$$

$$x_{23}=x_3-x_2$$

$$y_{23}=y_3-y_2$$

Substitute in the first differences.

$$x=t^3\,(x_{01}-2\,x_{12}+x_{23})+3\ t^2\,(x_{12}-x_{01})+3\ t\ x_{01}+x_0$$

$$y=t^3\,(y_{01}-2\,y_{12}+y_{23})+3\ t^2\,(y_{12}-y_{01})+3\ t\ y_{01}+y_0$$

These are the second differences.

$$x_{012}=x_{12}-x_{01}$$

$$y_{012}=y_{12}-y_{01}$$

$$x_{123}=x_{23}-x_{12}$$

$$y_{123}=y_{23}-y_{12}$$

Substitute in the second differences.

$$x=t^3\,(x_{123}-x_{012})+3\ t^2\ x_{012}+3\ t\ x_{01}+x_0$$

$$y=t^3\,(y_{123}-y_{012})+3\ t^2\ y_{012}+3\ t\ y_{01}+y_0$$

These are the third differences.

$$x_{0123}=x_{123}-x_{012}$$

$$y_{0123}=y_{123}-y_{012}$$

Substitute in the third differences.

$$x=t^3\ x_{0123}+3\ t^2\ x_{012}+3\ t\ x_{01}+x_0$$

$$y=t^3\ y_{0123}+3\ t^2\ y_{012}+3\ t\ y_{01}+y_0$$

The fastest way to evaluate a Bézier curve is the following:

$$b=x_{012}+x_{012}+x_{012}$$

$$c=x_{01}+x_{01}+x_{01}$$

$$e=y_{012}+y_{012}+y_{012}$$

$$f=y_{01}+y_{01}+y_{01}$$

$$x=((x_{0123}\,t+b)\,t+c)\,t+x_0$$

$$y=((y_{0123}\,t+e)\,t+f)\,t+y_0$$

The following is the TeX source for the previous equations. Tex is not as easy to use as PostScript, but I can not do equations in PostScript yet. I hope to work on a project to do equations in PostScript unless somebody has put equations in PostScript.

```
This is the definition of a B\'ezier curve.  When $t=0\,
 then x=x_{0}\, and\, y=y_{0}$.
$$x =a\, \,t^{3} +b\, \,t^{2} +c\, \,t +x_{0}$$
$$y =d\, \,t^{3} +e\, \,t^{2} +f\, \,t +y_{0}$$
When $t=1\, then x=x_{3}\, and\, y=y_{3}$.
$$x_{3} =a\, \,1^{3} +b\, \,1^{2} +c\, \,1 +x_{0}$$
$$y_{3} =d\, \,1^{3} +e\, \,1^{2} +f\, \,1 +y_{0}$$
Now solve for $a$ and $d$.
$$a = -b -c -x_{0} +x_{3}, \,d = -e -f -y_{0} +y_{3}$$
Substitute for $a$ and $d$.
$$x =b\, \,t^{2} \left( 1 -t\right) +c\, \,t \left( 1 -t^{2}\right)
 +t^{3} \left( x_{3} -x_{0}\right) +x_{0}$$
$$y =e\, \,t^{2} \left( 1 -t\right) +f\, \,t \left( 1 -t^{2}\right)
 +t^{3} \left( y_{3} -y_{0}\right) +y_{0}$$
Find the slope of $x$ and $y$.
$$ { dx\over dt}  =b\, \,t \left( 2 -3\,t\right) +c \left( 1 -3\,t^{2}
\right) +3\, \,t^{2} \left( x_{3} -x_{0}\right)$$
$$ { dy\over dt}  =e\, \,t \left( 2 -3\,t\right) +f \left( 1 -3\,t^{2}
\right) +3\, \,t^{2} \left( y_{3} -y_{0}\right)$$
Find the slope at the point $\{ x_{0},y_{0}\} $.
$$b\, \,0 \left( 2 -3*0\right) +c \left( 1 -3*0^{2}\right) +3*0^{2}
 \left( x_{3} -x_{0}\right) =c$$
$$e\, \,0 \left( 2 -3*0\right) +f \left( 1 -3*0^{2}\right) +3*0^{2}
 \left( y_{3} -y_{0}\right) =f$$
Point $\{ x_{1},y_{1}\} $ is on a line thru $\{ x_{0},y_{0}\} $
using the same slope and $t=1/3$.
$$x_{1} = { c\over \,3}  +x_{0}$$
$$y_{1} = { f\over \,3}  +y_{0}$$
Solve for $c$ and $f$.
$$c=3 \left( x_{1}-x_{0}\right), \,f =3 \left( y_{1}-y_{0}\right)$$
Substitute for $c$ and $f$.
$$x =b\, \,t^{2} \left( 1 -t\right) +t^{3} \left( 2\,x_{0} -3\,x_{1}
 +x_{3}\right) +3\, \,t \left( x_{1} -x_{0}\right) +x_{0}$$
$$y =e\, \,t^{2} \left( 1 -t\right) +t^{3} \left( 2\,y_{0} -3\,y_{1}
 +y_{3}\right) +3\, \,t \left( y_{1} -y_{0}\right) +y_{0}$$
Find the slope of $x$ and $y$.
$$ { dx\over dt}  =b\, \,t \left( 2 -3\,t\right) +3\, \,t^{2} \left(
 2\,x_{0} -3\,x_{1} +x_{3}\right) +3 \left( x_{1} -x_{0}\right)$$
$$ { dy\over dt}  =e\, \,t \left( 2 -3\,t\right) +3\, \,t^{2} \left(
 2\,y_{0} -3\,y_{1} +y_{3}\right) +3 \left( y_{1} -y_{0}\right)$$
Find the slope at the point $\{ x_{3},y_{3}\} $.
$$b\, \,1 \left( 2 -3*1\right) +3*1^{2} \left( 2\,x_{0} -3\,x_{1}
 +x_{3}\right) +3 \left( x_{1} -x_{0}\right) =3 \left( x_{0}
-2\,x_{1} +x_{3}\right) -b$$
$$e\, \,1 \left( 2 -3*1\right) +3*1^{2} \left( 2\,y_{0} -3\,y_{1}
 +y_{3}\right) +3 \left( y_{1} -y_{0}\right) =3 \left( y_{0}
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-2\,y_{1} +y_{3}\right) -e$$
Point $\{ x_{2},y_{2}\} $ is on a line thru $\{ x_{3},y_{3}\} $
using the same slope and $t=1/3$.
$$x_{3} = { 3\left(x_{0}-2\,x_{1}+x_{3}\right)-b\over \,3} +x_{2}$$
$$y_{3} = { 3\left(y_{0}-2\,y_{1}+y_{3}\right)-e\over \,3} +y_{2}$$
Solve for b and e.
$$b =3 \left(x_{0}-2\,x_{1}+x_{2}\right), \,e =3 \left(y_{0}-2\,y_{1}+y_{2}
\right)$$
Substitute for b and e.
$$x = -t^{3} \left(x_{0} -3\,x_{1} +3\,x_{2} -x_{3}\right) +3\,
\,t^{2} \left(x_{0} -2\,x_{1} +x_{2}\right) +3\, \,t \left(
x_{1} -x_{0}\right) +x_{0}$$
$$y = -t^{3} \left(y_{0} -3\,y_{1} +3\,y_{2} -y_{3}\right) +3\,
\,t^{2} \left(y_{0} -2\,y_{1} +y_{2}\right) +3\, \,t \left(
y_{1} -y_{0}\right) +y_{0}$$
This is the B\'ezier curve in terms of the control points. $\left[
\{ x_{0},y_{0}\},\{ x_{1},y_{1}\},\{ x_{2},y_{2}\},\{ x_{3},y_{3}\} \right]$.

Now I will simplify the calculations using a difference table.
These are the first differences.
$$x_{01} =x_{1} -x_{0}$$
$$y_{01} =y_{1} -y_{0}$$
$$x_{12} =x_{2} -x_{1}$$
$$y_{12} =y_{2} -y_{1}$$
$$x_{23} =x_{3} -x_{2}$$
$$y_{23} =y_{3} -y_{2}$$
Substitute in the first differences.
$$x =t^{3} \left(x_{01} -2\,x_{12} +x_{23}\right) +3\, \,t^{2}
\left(x_{12} -x_{01}\right) +3\, \,t\, \,x_{01} +x_{0}$$
$$y =t^{3} \left(y_{01} -2\,y_{12} +y_{23}\right) +3\, \,t^{2}
\left(y_{12} -y_{01}\right) +3\, \,t\, \,y_{01} +y_{0}$$
These are the second differences.
$$x_{012} =x_{12} -x_{01}$$
$$y_{012} =y_{12} -y_{01}$$
$$x_{123} =x_{23} -x_{12}$$
$$y_{123} =y_{23} -y_{12}$$
Substitute in the second differences.
$$x =t^{3} \left(x_{123} -x_{012}\right) +3\, \,t^{2}\, \,x_{012}
+3\, \,t\, \,x_{01} +x_{0}$$
$$y =t^{3} \left(y_{123} -y_{012}\right) +3\, \,t^{2}\, \,y_{012}
+3\, \,t\, \,y_{01} +y_{0}$$
These are the third differences.
$$x_{0123} =x_{123} -x_{012}$$
$$y_{0123} =y_{123} -y_{012}$$
Substitute in the third differences.
$$x =t^{3}\, \,x_{0123} +3\, \,t^{2}\, \,x_{012} +3\, \,t\, \,x_{01}
+x_{0}$$
$$y =t^{3}\, \,y_{0123} +3\, \,t^{2}\, \,y_{012} +3\, \,t\, \,y_{01}
+y_{0}$$

The fastest way to evaluate a B\'ezier curve is the following:
$$b =x_{012}+x_{012}+x_{012}$$

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$$c =x_{01}+x_{01}+x_{01}$$
$$e =y_{012}+y_{012}+y_{012}$$
$$f =y_{01}+y_{01}+y_{01}$$
$$x =\left( \left( x_{0123} \,t +b\right) t +c\right) t +x_{0}$$
$$y =\left( \left( y_{0123} \,t +e\right) t +f\right) t +y_{0}$$
\bye
```