

T_EX source

The length of a Berier curve is an elliptic integral. This is a Bezier curve.

$$x = a_3 t^3 + a_2 t^2 + a_1 t + x_0$$

$$y = b_3 t^3 + b_2 t^2 + b_1 t + y_0$$

The control points of a Bezier curve are:

$$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2) \quad (x_3, y_3)$$

This is the curve for those control points.

$$a_1 = 3(x_1 - x_0) \quad a_2 = 3(x_0 - 2x_1 + x_2) \quad a_3 = -x_0 + 3x_1 - 3x_2 + x_3$$

$$b_1 = 3(y_1 - y_0) \quad b_2 = 3(y_0 - 2y_1 + y_2) \quad b_3 = -y_0 + 3y_1 - 3y_2 + y_3$$

The length of the Bezier curve from 0 to u is:

$$\int_0^u \sqrt{(a_1 + 2a_2 t + 3a_3 t^2)^2 + (b_1 + 2b_2 t + 3b_3 t^2)^2} dt$$

The control points of an example Bezier curve are:

$$(x_0 = 6, y_0 = 8) \quad (x_1 = 1, y_1 = 10) \quad (x_2 = 7, y_2 = 3) \quad (x_3 = 4, y_3 = 4)$$

Solve for $a_1, a_2, a_3, b_1, b_2, b_3$.

$$a_1 = -15 \quad a_2 = 33 \quad a_3 = -20 \quad b_1 = 6 \quad b_2 = -27 \quad b_3 = 17$$

The Bezier curve for this example is:

$$x = -20t^3 + 33t^2 - 15t + 6 \quad y = 17t^3 - 27t^2 + 6t + 8$$

For this example the length is:

$$3 \int_0^1 \sqrt{689t^4 - 1492t^3 + 1076t^2 - 292t + 29} dt$$

Numerical integration gives 7.237223368328592885826619210956022413967067986017504163362886516

DERIVE factored the polynomial:

$$\begin{aligned} & 689t^4 - 1492t^3 + 1076t^2 - 292t + 29 = \\ & 689 \left(t + \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} + i \left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} + \frac{7}{689} \right) \right) \\ & \left(t + \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} - i \left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} + \frac{7}{689} \right) \right) \\ & \left(t - \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721}} - \frac{373}{689} + i \left(\sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} - \frac{7}{689} \right) \right) \end{aligned}$$

$$\left(t - \sqrt{\frac{\sqrt{6005}}{1378} + \frac{23377}{474721} - \frac{373}{689}} + i \left(\frac{7}{689} - \sqrt{\frac{\sqrt{6005}}{1378} - \frac{23377}{474721}} \right) \right)$$

Now define values k, m, a, b :

$$k = 3\sqrt{689} \quad m = [1, 1, 1, 1] \quad b = [1, 1, 1, 1] \quad e = [e_1, e_2, e_3, e_4]$$

$$a = [-0.216589 + 0.0937743 i, -0.216589 - 0.0937743 i, -0.866139 + 0.0734550 i, -0.866139 - 0.0734550 i]$$

Do a partial fraction expansion.

$$\begin{aligned} I(m) &= D219(1, m, 4, a, b, e) \\ I(m) &= I(4e_1) + (1.29909 + 0.375097 i) I(3e_1) + (0.383342 + 0.365466 i) I(2e_1) \\ &\quad - (0.0228475 - 0.0784923 i) I(e_1) \end{aligned}$$

Reduce $I(4e_1)$.

$$\begin{aligned} R35(0, 1, 4, a, b, e) &= 0 \\ A(2e_1 + e_2 + e_3 + e_4) + 3I(4e_1) + (3.24774 + 0.937743 i) I(3e_1) + (0.766684 + 0.730933 i) I(2e_1) \\ &\quad - (0.0342712 - 0.117738 i) I(e_1) = 0 \\ A(2e_1 + e_2 + e_3 + e_4) &= -0.138815 + 0.00794130 i \end{aligned}$$

Substitute $I(4e_1)$.

$$\begin{aligned} I(m) &= (0.216516 + 0.0625162 i) I(3e_1) + (0.127780 + 0.121822 i) I(2e_1) \\ &\quad - (0.0114237 - 0.0392461 i) I(e_1) + 0.0462716 - 0.00264710 i \end{aligned}$$

Reduce $I(3e_1)$.

$$\begin{aligned} R35(-1, 1, 4, a, b, e) &= 0 \\ A(e_1 + e_2 + e_3 + e_4) + 2I(3e_1) + (1.94864 + 0.562645 i) I(2e_1) \\ &\quad + (0.383342 + 0.365466 i) I(e_1) - (0.0114237 - 0.0392461 i) I(0) = 0 \\ A(e_1 + e_2 + e_3 + e_4) &= -0.0846852 \end{aligned}$$

Substitute $I(3e_1)$.

$$\begin{aligned} I(m) &= -0.0655894 I(2e_1) - (0.0415000 + 0.0123012 i) I(e_1) \\ &\quad + (0.00246348 - 0.00389163 i) I(0) + 0.0554395 \end{aligned}$$

Reduce $I(2e_1)$.

$$\begin{aligned} R35(-2, 1, 4, a, b, e) &= 0 \\ A(e_2 + e_3 + e_4) + I(2e_1) + (0.0114237 - 0.0392461 i) I(-e_1) \\ &\quad + (0.649549 + 0.187548 i) I(e_1) = 0 \\ A(e_2 + e_3 + e_4) &= -0.949300 - 0.327220 i \end{aligned}$$

Substitute $I(2e_1)$.

$$\begin{aligned} I(m) &= (0.000749279 - 0.00257413 i) I(-e_1) + 0.00110358 I(e_1) \\ &\quad + (0.00246348 - 0.00389163 i) I(0) - 0.00682448 - 0.0214622 i \end{aligned}$$

Calculate the basic integrals numerically.

$$I(-e_1) = -17.2709 + 33.5142 i \quad I(0) = 12.0414 \quad I(e_1) = -3.86310 - 1.12917 i$$

Substitute the values of the integrals.

$$I(m) = 0.0919054 \quad k I(m) = 7.23722$$

That is a good match.

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